

CLASSICAL AND QUANTUM TWO-BODY PROBLEM IN GENERAL RELATIVITY¹.

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ABSTRACT. The two-body problem in general relativity is reduced to the problem of an effective particle (with an energy-dependent relativistic reduced mass) in an external field. The effective potential is evaluated from the Born diagram of the linearized quantum theory of gravity. It reduces to a Schwarzschild-like potential with two different ‘Schwarzschild radii’. The results derived in a weak field approximation are expected to be relevant for relativistic velocities.

1. In both non-relativistic and special relativistic mechanics, classical and quantum, the two-body problem for (spinless) point particles is reduced to the conceptually simpler problem of a single effective particle moving in an external field. The only exception to this picture so far seems to be the general theory of relativity, where the two-body problem has been treated in a considerably more complicated way: as a field-theoretic problem with singularities [1, 2] (or as a problem of finite size bodies interacting with a gravitational field [31]). Here we propose to treat gravitational two-particle interaction in much the same way as electromagnetic interactions have been tackled previously [4, 5] in the quasipotential approach [6] which found its natural place in the constraint Hamiltonian framework of References [7] and [8].¹ Unlike other first-order (in $1/c^2$) semi-relativistic treatments (based on a quantum field theoretic derivation of the two-particle potential) [12], our approach is fully relativistic. Here we shall consider the two-body problem in the leading order of perturbation

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¹A similar approach is being developed by a number of authors (see, e.g., References [9] and [10]); the reader will find a comprehensive bibliography in References [11] and [8].

theory in G , the Newtonian gravitational constant. It is reduced to the problem of an effective particle (with an energy-dependent relativistic reduced mass) in an external Schwarzschild-like field with two different ‘Schwarzschild radii’, in g_{00} and g_{ij} respectively.

2. We shall briefly summarize the constraint Hamiltonian approach to the relativistic two-body problem and will introduce the notion of an effective particle in this approach.

We define the generalized two-point (spinless) particle mass shell as a 14-dimensional sub-manifold of the 16-dimensional ‘large phase space’ Γ of Minkowski space co-ordinates x_l , x_2 and four-momenta p_1, p_2 , given by two first-class constraints. We postulate (as in [4, 5]) that the difference $p_1^2 - p_2^2$ is independent of the interaction:

$$\varphi = \frac{1}{2}(m_1^2 + p_1^2 - m_2^2 - p_2^2) = pP = 0, \quad (0.1)$$

where m_1, m_2 are the masses of the two particles, P and p are the total and the relative momenta:

$$\begin{aligned} P &= p_1 + p_2, \quad p = \mu_1 p_2 - \mu_2 p_1, \quad \mu_1 + \mu_2 = 1, \\ \mu_1 - \mu_2 &= \frac{m_1^2 - m_2^2}{w^2}, \quad w^2 = -P^2 (> 0). \end{aligned} \quad (0.2)$$

(We are using the space-like signature $-+++$ for the metric tensor.)

The non-relativistic reduced mass m is defined by the equation $mM = m_1 m_2$, where $M = m_1 + m_2$ is the total mass. We use the same equation to define the relativistic reduced mass m_w , just replacing M by the total relativistic mass $w (= (-P^2)^{1/2})$:

$$m_w = \frac{m_1 m_2}{w}. \quad (0.3)$$

The effective particle four-momentum P_{eff} is then defined in the centre-of-mass frame (in which $P = (w, \mathbf{0}), p = (0, \mathbf{p})$) by

$$p_{\text{eff}} = (E, \mathbf{p}), \quad E = (m_w^2 + b^2(w))^{1/2} = \frac{w^2 - m_1^2 - m_2^2}{2w}, \quad (0.4)$$

where $b^2(w)$ is the one-shell value of the relative momentum square

$$b^2(w) = \frac{w^4 - 2(m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2}{4w^2}. \quad (0.5)$$

In the first approximation in the coupling constants (charges) $e_{1,2}$, the electromagnetic interaction of two charged particles has been given by the Hamiltonian constraint [4, 5, 8]

$$\begin{aligned} H_{\text{Coul}} &= \frac{1}{2}[m_w^2 + \mathbf{p}^2 - (E - V_{\text{Coul}})^2] = 0, \quad \mathbf{p}^2 = p^2 = p_{\text{eff}}^2 + \frac{(Pp_{\text{eff}})^2}{w^2}, \\ V_{\text{Coul}} &= \frac{e_1 e_2}{4\pi r}, \quad r = (x_{\perp}^2)^{1/2} = \left(x^2 + \frac{(xP)^2}{w^2}\right)^{1/2}, \quad x = x_1 - x_2. \end{aligned} \quad (0.6)$$

(Note that the constraint (6) is manifestly a Poincare invariant; no semi-relativistic approximation of the type of the $1/c^2$ expansion has been made.) The idea of the present note is to describe in a similar fashion the gravitational interaction of two relativistic masses by setting

$$H = H_{\text{Grav}} = \frac{1}{2}[m_w^2 + g^{\mu\nu} P_{\text{eff}\mu} P_{\text{eff}\nu}] = 0, \quad (0.7)$$

where $g^{\mu\mu}$ is some appropriate modification of the Schwarzschild metric.

3. The actual computation of the electromagnetic Hamiltonian constraint (which includes corrections to H_{Coul}) has been effected in the quasipotential approach to quantum electrodynamics [4, 5]. We shall pursue here a similar path starting with a standard linearized form of quantum gravity (cf. References [13]).

According to Dirac's general theory [14], the quantum counterpart of the first-class constraint (7) is the relativistic 'Schrödinger equation'

$$\left[m_w^2 + \frac{1}{6}R - |g|^{-1/2} \partial_\mu (|g|^{1/2} g^{\mu\nu}) \partial_\nu \right] \Psi = 0 \quad (0.8)$$

for the state vector $\Psi(x)$. Here R is the scalar curvature. (As pointed out by Penrose [15], the $R/6$ term is necessary in order to ensure conformal invariance of the zero mass limit.) [The Laplace-Beltrami operator provides the appropriate generally covariant ordering of the canonical variables $x_{\text{eff}} (= -(Px)w^{-2}P^\mu + x_\perp^\mu)$ and $p_{\text{eff}\mu} = -i\partial_\mu$.] The momentum space counterpart of (8) is to be identified with the local quasipotential equation [4, 5] (written here in the centre of the mass frame)

$$\begin{aligned} G_w^{-1}(\mathbf{p}) \tilde{\Psi}(\mathbf{p}) + (V_* \tilde{\Psi})(\mathbf{p}) \\ \equiv 2w[\mathbf{p}^2 - b^2(w)] \tilde{\Psi}(\mathbf{p}) + \int V(\mathbf{p}, \mathbf{q}) \frac{d^3q}{(2\pi)^3} = 0, \end{aligned} \quad (0.9)$$

the potential V is determined order by order in G from the Lippmann-Schwinger-type equation:

$$T + V + V_* G_w T = 0, \quad G_w(\mathbf{k}) = [2w(\mathbf{k} - b^2(w) - i0)^{-1}] \quad (0.10)$$

and from the Feynman expansion of the scattering amplitude $T = T_w(\mathbf{p}, \mathbf{q})$ in a quantum theory of gravitationally interacting scalar particles.

We shall treat Equations (8) and (9) in the leading order approximation of perturbation theory. The linearized form of (8) is obtained by setting

$$g_{\mu\nu} = \eta_{\nu\mu} + h_{\nu\mu}, \quad \eta_{\nu\mu} = \text{diag}(-1, +1, +1, +1), \quad |h_{\nu\mu}| \ll 1, \quad (0.11)$$

and using $g^{\mu\nu} \approx \eta^{\nu\mu} - h^{\nu\mu}$, $|g| \approx 1 + h^\mu_\mu$, $R \approx \partial_\mu \partial_\nu h^{\mu\nu} - \square h^\mu_\mu$ (where Lorentz indices are raised and lowered by η). Up to terms of order $0(h^2)$ Equation (8) reads:

$$\left\{ m_w^2 - \square + [h^{\mu\nu} \partial_\mu \partial_\nu + \frac{1}{6}(\partial_\mu \partial_\nu h^{\mu\nu}) + (\partial_\mu h^{\mu\nu} - \frac{1}{2} \partial^\nu h^\lambda_\lambda) \partial_\nu - \frac{1}{6}(\square h^\mu_\mu)] \right\} \Psi = 0 \quad (0.12)$$

Thus, in the leading order of perturbation theory we have:

$$\begin{aligned} & \tilde{h}^{\mu\nu}(p-q) q_\mu q_\nu + \frac{1}{6}(p-q)_\mu (p-q)_\nu \tilde{h}^{\mu\nu}(p-q) \\ & + [(p-q)_\mu \tilde{h}^{\mu\nu}(p-q) - \frac{1}{2}(p-q)^\nu \tilde{h}^\lambda_\lambda(p-q)] q_\nu \\ & - \frac{1}{6}(p-q)^2 \tilde{h}^\mu_\mu(p-q) = \frac{1}{2w} T_w^{(1)}(\mathbf{p}, \mathbf{q}), \\ & p = (p_\mu) = (-E, \mathbf{p}), \quad q = (-E, \mathbf{q}), \end{aligned} \quad (0.13)$$

where E is given by (4) (and $\mathbf{p}^2 = \mathbf{q}^2 = b^2(w)$ (on the mass shell)).

4. The Born approximation $T_w^{(1)}$ for the two-particle scattering amplitude is derived from the Lagrangian density:

$$\mathcal{L} = -|g|^{1/2} \left[\frac{R}{16\pi G} + \frac{1}{2} \sum_{k=1,2} (g^{\mu\nu} \partial_\mu \Phi_k \partial_\nu \Phi_k + m_k^2 \Phi_k^2 + \frac{1}{6} R \Phi_k^2) \right] \quad (0.14)$$

in the weak field approximation (11).² The expression (14) differs by the $\frac{1}{6} R \Phi_k^2$ term from the Lagrangian used in References [13] (corresponding to the $R/6$ in the Schrödinger Equation (8)). The one-graviton exchange diagram between particles 1 and 2 gives

$$T_w^{(1)}(\mathbf{p}, \mathbf{q}) = 4\pi G \Gamma_{\kappa\lambda}^{(1)} \tilde{\mathcal{D}}^{\kappa\lambda, \mu\nu} (p^{(1)} - q^{(1)}) \Gamma_{\mu\nu}^{(2)}, \quad (0.15)$$

$$\Gamma_{\mu\nu}^{(k)} = i[p_\mu^{(k)} q_\nu^{(k)} + p_\nu^{(k)} q_\mu^{(k)} - \eta_{\mu\nu} (p^{(k)} q^{(k)} + \frac{1}{3} (p^{(k)} - q^{(k)})^2 + m_k^2) + \frac{1}{3} (p^{(k)} - q^{(k)})_\mu (p^{(k)} - q^{(k)})_\nu], \quad k = 1, 2, \quad (0.16)$$

$$\tilde{\mathcal{D}}^{\kappa\lambda, \mu\nu}(k) = \frac{\eta^{\kappa\mu} \eta^{\lambda\nu} + \eta^{\kappa\nu} \eta^{\lambda\mu} - \eta^{\kappa\lambda} \eta^{\mu\nu}}{k^2 - i0}, \quad (0.17)$$

$$p^{(1)} = (E_1, \mathbf{p}), \quad p^{(2)} = (E_2, -\mathbf{p}), \quad q^{(1)} = (E_1, \mathbf{q}), \quad q^{(2)} = (E_2, -\mathbf{q}), \quad E_k = \mu_k w. \quad (0.18)$$

Inserting (16), (17) and (18) into Equation (15), we obtain:

$$T_w^{(1)}(\mathbf{p}, \mathbf{q}) = 16\pi G \left[\frac{2E^2 w^2 - m_1^2 m_2^2}{(\mathbf{p} - \mathbf{q})^2} - Ew - \frac{m_1^2 + m_2^2}{6} + \frac{1}{12} (\mathbf{p} - \mathbf{q})^2 \right]. \quad (0.19)$$

5. The next step is to evaluate $h_{\mu\nu}$ from Equations (13) and (19). To this end we shall use the Euclidean invariant 'stationary gauge' in which

$$h_{0i} = 0, \quad h_{00} = \frac{r_t}{r} \quad (r_t = \text{const}), \quad h_{ij} = B(r) x_i x_j \quad (\text{for } x \neq 0). \quad (0.20)$$

(The last condition means that we require the angular part of ds^2 to have its flat space form $r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$, which is the standard co-ordinate choice for the Schwarzschild solution.) This amounts to setting

$$h_{00}(\mathbf{p} - \mathbf{q}) = \frac{4\pi r_t}{(\mathbf{p} - \mathbf{q})^2}$$

$$h_{ij}(\mathbf{p} - \mathbf{q}) = 4\pi r_s \frac{(\mathbf{p} - \mathbf{q})^2 \delta_{ij} - 2(p_i - q_i)(p_j - q_j)}{(\mathbf{p} - \mathbf{q})^4} + C \delta_{ij}, \quad (0.21)$$

where r_t, r_s and C are constants of the motion. Inserting (19) and (21) into Equation (13), we find³

$$r_t = 2Gw \left[1 - \frac{4b^2}{m_w^2} \left(2\frac{E}{w} - 3\frac{b^2}{w^2} \right) \right], \quad (0.22)$$

$$r_s = 2Gw \left[1 + \frac{4b^2}{m_w^2} \left(2\frac{E}{w} - 3\frac{b^2}{w^2} \right) \right], \quad C = -\frac{8\pi G}{w};.$$

²The naive G -perturbation theory of (14) is nonrenormalizable. According to the general discussion in Reference [16], Equation (14) gives a correct description of gravitational interactions only on tree-graph level and at a relatively low energy scale (much less than 10^{19} GeV for elementary particles). In order to compute V consistently to arbitrary orders in G from Equation (10) one should use a nontrivial renormalizable extension of (14) if there is any (at present only extended supergravity is a hopeful candidate).

³*The expression for r_s does not coincide with the correct semi-relativistic approximation of Reference [1]. The results of Reference [1,2] indicate that the agreement will be restored if one takes into account the semi-relativistic contribution to the effective potential coming from the Feynman diagrams of order G^2 .

Thus, we end up with the following x -space expression for the metric tensor:

$$g_{00} = -\left(1 - \frac{r_t}{r}\right), \quad g_{0i} = 0, \quad g_{ij} = \delta_{ij} + r_s \frac{x_i x_j}{r^3} - \frac{8\pi G}{w} \delta_{ij} \delta(\mathbf{x}). \quad (0.23)$$

The last (δ -function) term does not contribute to the classical motion and will be ignored in the sequel (it may only be relevant, for a quantum s-wave effect). Clearly, in the text body limit, i.e., for $(m_1 + m_2)^2 \gg m_1 m_2$, and for $|w(m_1 + m_2)^{-1} - 1| \ll 1$ (slow motion), the right-hand sides of Equations (23) go into the linearized Schwarzschild solution (both r_t and r_s tending to the Schwarzschild radius $2(m_1 + m_2)G$).

6. We are now prepared to treat the classical gravitational two-body problem by inserting the metric (23) into the Hamiltonian constraint (7). Going to spherical co-ordinates, we can rewrite Equation (7) in the form

$$H = \frac{1}{2} \left[m_w^2 - \left(1 - \frac{r_t}{r}\right)^{-1} p_0^2 + \left(1 - \frac{r_s}{r}\right) p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{\sin^2 \theta} p_\varphi^2 \right] \approx 0 \quad (0.24)$$

A standard computation using the initial condition $\theta = \pi/2, p_\theta = 0$ (cf. [17]) gives:

$$-p_0 = E \quad (\dot{E} = 0), \quad p_\varphi = J \quad (\dot{J} = 0), \quad p_r^2 \left(1 - \frac{r_t}{r}\right) = b^2 + E^2 \frac{r_t}{r} \left(1 - \frac{r_t}{r}\right)^{-1} - \frac{J^2}{r^2}. \quad (0.25)$$

Introducing the radius variable $u = r^{-1}$ and setting $du/d\varphi = u'$ we obtain

$$J^2(u'^2 + u^2 - r_s u^3) + [r_s b^2 - r_t E^2 (1 - r_s u)(1 - r_t u)^{-1}] u' = b^2. \quad (0.26)$$

We look for a solution of this equation of the form

$$u = \ell^{-1} [1 + \epsilon(\cos \eta\varphi + f(\eta\varphi))], \quad (0.27)$$

where the natural dimensionless small parameter is now r/ℓ . The unknown function $f(\varphi)$ is expected to be a small correction (of order r/ℓ) to the Schwarzschild-like solution. Inserting in (26) and comparing the coefficients of $\cos 2\eta\varphi$, $\cos \eta\varphi$, and the constant term, we find:

$$\begin{aligned} \eta &= 1 - \frac{3r_t}{2\ell} + \frac{r_t - r_s}{2\ell}, \quad \epsilon^2 = 1 + \frac{4r_t}{\ell} + \frac{\ell^2 b^2}{J^2} \left(1 + \frac{3r_t + r_s}{\ell}\right), \\ \ell &= 2J^2(r_t m_w^2 - r_s b^2)^{-1} + O(r_s). \end{aligned} \quad (0.28)$$

The terms containing f , f' and $\cos^3 \eta\varphi$ lead to the differential equation

$$\sin \eta\varphi \cdot f' - \cos \eta\varphi \cdot f + \frac{\epsilon r_s}{2\ell} \cos^3 \eta\varphi = 0. \quad (0.29)$$

Its solution, satisfying $f < |\cos \eta\varphi|$ for all φ is

$$f = \frac{\epsilon r_s}{2\ell} (1 - |\sin \eta\varphi|)^2 + O\left[\left(\frac{r_s}{\ell}\right)^2\right] \quad (0.30)$$

(which is of order r_s/ℓ in accord with our expectation).

The solution (27), (28) and (30) so obtained, reduces to the classical one [1] in the semirelativistic and test body limit.⁴ It is expected to give a more accurate description of the two-particle motion for relativistic velocities and weak gravitational forces.

⁴This is to be contrasted with the results of some previous first-order (in G) relativistic approaches [18], which give incorrect values for η even in the test body limit.

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REFERENCES

1. Einstein, A., Infeld, L., and Hoffmann, B., *Ann. Math.* 39, 65 (1938); 41, 455 (1940); Infeld, L. and Plebanski, J., *Motion and Relativity*, Pergamon Press, Oxford and PWN, Warszawa, 1960 (see, in particular, Section 3 of Chapter V).
2. Infeld, L. and Michalska-Trautman, R., *Ann. Phys. (N. Y.)* 55, 561 (1969).
3. Petrova, N.M., *Zh. Eksp. Teor. Fiz.* 19, 989 (1949) (in Russian).
4. Todorov, I.T., *Phys. Rev. D* 3, 2351 (1971); see also *Properties of Fundamental Interactions*, ed. A. Zichichi, Editrice Compositori, Bologna, 1973, Vol. 9C, pp. 951-979.
5. Rizov, V.A. and Todorov, I.T., *Sov. J. Part. Nucl.* 6, 269 (1975); Rizov, V.A., Todorov, I.T., and Aneva, B.L., *Nucl. Phys. B* 98, 447 (1975).
6. Logunov, A.A. and Tavkhelidze, A.N., *Nuovo Cimento* 29, 380 (1963); Logunov, A.A. et al., *Nuovo Cimento* 30, 134 (1963).
7. Todorov, I.T., 'Dynamics of Relativistic Point Particles as a Problem with Constraints', *Commun. JINR E2-10125*, Dubna (1976); Molotkov, V.V. and Todorov, I.T., 'Gauge Dependence of World Lines and Invariance of the S-matrix in Relativistic Point Particle Dynamics', *ICTP, Trieste, Internal Report IC/80/101*; *Commun. Math. Phys.* 79, 111 (1981).
8. Todorov, I.T., 'Constraint Hamiltonian Dynamics of Directly Interacting Relativistic Point Particles', Lectures presented at the Summer School in Mathematical Physics, Bogazici University, Bebek, Istanbul (1979) and at the XVIIth Winter School of Theoretical Physics, Karpacz (1980) (to be published).
9. Droz-Vincent, Ph., *Ann. Inst. H Poincaré* 27, 407 (1977); 'N-body Relativistic Systems', *College de France preprint*, Paris (1979); Bel, L., and Martin, J., *Ann. Inst. IL Poincaré* 22A, 173 (1975); Fustero, F.X. and Lapedra, R., *Phys. Rev. D* 17, 2821 (1978).
10. Dominici, D., Gomis, J., and Longhi, G., *Nuovo Cimento* 48A, 257 (1978); 48B, 152 (1978); Giachetti, R. and Sorace, E. 'Relativistic Two-Body Interactions: A Hamiltonian Formulation' (to appear in *Nuovo Cimento*).
11. Takabayasi, T., 'Relativistic Mechanics of Confined Particles as Extended Model of Hadrons – The Bifocal Case', *Prog. Theor. Phys. Suppl. No. 67*, 1 (1979).
12. Gupta, S. and Radford, S., *Phys. Rev. D* 21, 2213 (1980) and references therein.
13. Gupta, S., *Proc. Phys. Soc.* 65A, 161, 608 (1952); Corinaldesi, E., *Proc. Phys. Soc.* 69A, 189 (1956).
14. Dirac, P.A.M., *Lectures on Quantum Mechanics*, Belfer Grad. School of Science, Yeshiva Univ., N.Y., 1964.
15. Penrose, R., in *Relativity, Groups and Topology*, eds. C.M. de Witt, and B. de Witt, Gordon and Breach, N.Y., 1964.
16. Weinberg, S., in *Gravitational Theories Since Einstein*, eds. S. Hawking and W. Israel, Cambridge Univ. Press, Cambridge, 1979.
17. Weinberg, S., *Gravitation and Cosmology*, Wiley, N.Y., 1972.

18. Havas, P. and Goldberg, J., Phys. Rev., 128, 398 (1962).

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